TOPOLOGICAL RECURSION AND ELSV FOR DOUBLE HURWITZ

joint work with
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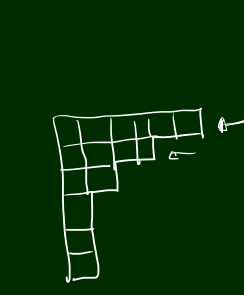
① Definition of H



$D \geq 1$ $\quad \lambda \vdash D \quad \lambda_1 \geq \dots \geq \lambda_\ell > 0$

$|\lambda| = D \ , \ \ell(\lambda) = \ell$ length

$\# \mathrm{Aut}\,\lambda = \prod_{i \geq 1} m_i(\lambda)! \quad m_i(\lambda) = \#\{\text{parts of } \lambda = i\}$.

C_λ conjugacy class in $\mathfrak{S}_D$

$Z\mathbb{Q}[\mathfrak{S}_D]$ has a basis given $\hat{C}_\lambda = \sum_{g \in C_\lambda} g$

$\lambda, \mu \vdash D$, $k \geq 0$

$$H^{\bullet, k}_{\lambda,\mu} = \frac{1}{D!} [\mathrm{id}]\ \hat{C}_\lambda \frac{\hat{C}^k_{(2,1,\dots,1)}}{k!} \hat{C}_\mu$$

$$= \frac{1}{D!}\frac{1}{k!} \# \left\{ \begin{matrix} \sigma_0, \sigma_\infty, \tau_1, \dots, \tau_k \in \mathfrak{S}_D \\ \sigma_0 \in C_\lambda,\ \sigma_\infty \in C_\mu \\ \tau_i \text{ transpositions} \end{matrix} \ \middle| \ \sigma_0 \tau_1 \dots \tau_k \sigma_\infty = \mathrm{id} \right\}$$

$$= \frac{1}{D!}\frac{1}{k!} \# \left\{ \begin{matrix} \rho \in \mathrm{Hom}(\pi_1(\mathbb{S}^2 - \{0,\infty,p_1,\dots,p_k\}), \mathfrak{S}_D) \\ \text{such that } \rho(\gamma_0) \in C_\lambda,\ \rho(\gamma_\infty) \in C_\mu \\ \rho(\gamma_i) \text{ transpositions} \end{matrix} \right\}$$

$$= \frac{1}{k!} \sum_f{}' \frac{1}{\# \mathrm{Aut} f}$$

counting branched covers

$f : \Sigma \to \mathbb{S}^2$
branched cover
ramified above $\underset{\lambda}{0}$ and $\underset{\mu}{\infty}$
(maybe disconnected).

$g \geq 0$ $\quad [H^\circ_{\lambda,\mu}]_g$ inclusion-exclusion from $H^\bullet$
connected covers ($\mathrm{Im}\,\rho$ transitive)

Riemann-Hurwitz $\quad D \cdot 2 = 2 - 2g - \sum_i (\lambda_i - 1) - \sum_i (\mu_i - 1) - k$.

$$\boxed{2g - 2 + \ell(\lambda) + \ell(\mu) = k}$$

Simple H# : $\mu = (1,\dots,1)$ $\quad \infty$ not ramified.
d-orbifold H# $\quad \mu = (d,\dots,d)$
double H# : μ arbitrary

interested in computing for fixed g
and fixed $\ell(\lambda) = n$

- ✓ → Tau functions of 2d Toda hierarchy (Okounkov 00, Harnad, Guay-Paquet ~ 14).
- ✓ → GW($\mathbb{P}^1$) ⟷ Double Hurwitz # (Okounkov-Pandharipande).
  if one allow to replace $\frac{\hat{C}^k_{(2,1,\dots,1)}}{k!} \to \hat{B} \in Z\mathbb{Q}[\mathfrak{S}_D]$.
- ✔ → mirror symmetry : H# as periods of differentials on variety $\mathfrak{S}^n$
  (~ Bouchard-Mariño conj for simple H# 07)
- → ?? → ELSV formulas (2001). (expression via intersection on $\overline{\mathcal{M}}_{g,n}$).
- → ?? → ~07 topological recursion (Eynard, Orantin) : recursion on $2g-2+n$ to compute H#

② ELSV formula

Ekedahl, Lando, Shapiro, Vainshtein

$\mathbb{E}_g$ = Hodge bundle of hol 1-forms $\to \overline{\mathcal{M}}_{g,n} \ni [(C, p_1 \dots p_n)]$

$c_h(\mathbb{E}_g) = \Lambda^\vee(-1)$
$= \sum_{h=0}^{g} c_h(\mathbb{E}_g)(-1)^h$

$\psi_i = c_1(T^*_{p_i} C) \in H^2$
$i = 1, \dots, n$

Thm $$[H^\circ_{\lambda,\phi}]_g = \prod_{i=1}^{n} \frac{\lambda_i^{\lambda_i}}{\lambda_i!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda^\vee}{\prod_{i=1}^n (1 - \lambda_i \psi_i)}$$

quasipolynomiality behavior $\qquad$ polynomial in λ

- Application (Goulden, Jackson, Vakil)

$$\lambda = (D) \qquad 1 + \sum_{g \geq 1} t^{2g} \int_{\overline{\mathcal{M}}_{g,1}} \frac{\Lambda^\vee(-D)}{(1 - D\psi_1)} = \left(\frac{t/2}{\sinh t/2} \right)^{D+1}$$

Idea $\quad \mathcal{H}_{g,\lambda}$ moduli of $(C_g, p_1, \dots, p_n, z_1, \dots, z_k, \pi : C_g \to \mathbb{P}^1)$
$\mathrm{ord}_{p_i} \pi = -\lambda_i$ (poles).
$\mathrm{ord}_{z_i} d\pi = 1$ (future ramification points).

$$Z_{g,\lambda} : \mathcal{H}_{g,\lambda} \to \mathbb{C}^k / \mathbb{C}^*$$
$$(C, \pi) \mapsto [\pi(z_1), \dots, \pi(z_k)]$$

$$\frac{\deg \mathcal{H}_{g,\lambda}}{k!} = [H^\circ_{\lambda,\phi}]_g$$

$$\mathcal{H}_{g,\lambda} \xhookrightarrow{\Phi} \mathcal{P} \to \overline{\mathcal{M}}_{g,n}$$

principal parts of π at the poles.

$\mathcal{P}' = \{(\pi_1, \dots, \pi_n) \mid \sum_i \mathrm{Res}_{p_i}(\pi_i \omega) = 0 \ \ \forall \omega \in H^0(K_C)\}$.

compactify $\quad \overline{\mathcal{P}'} \to \overline{\mathcal{M}}_{g,n}$ cone.

extract the degree by Segre class computation.

③ Periods of (multi)differentials.

$$[H^\circ_{(D)}]_{g=0} = \frac{D^{D-2}}{D!} \quad \text{(Cayley tree enumeration).}$$

$$[H^\circ_{(\lambda_1,\lambda_2)}]_{g=0} = \frac{\lambda_1^{\lambda_2} \lambda_2^{\lambda_1}}{(\lambda_1+\lambda_2)\lambda_1!\lambda_2!} \quad \text{(cannot decorated by trees).}$$

$$\mathcal{S} = \{(X,y) \in \mathbb{C}^* \times \mathbb{C} \mid X = y e^{-y}\} \quad \text{mirror curve of } \mathbb{C}^3 \text{ framed } f \to \infty$$

disk $\to$ $\boxed{\omega_{0,1} = y \frac{dX}{X}}$ $\qquad \mathrm{Res}_{X=0} \frac{X^{-D}}{D} \omega_{0,1} = [H^\circ_{(D)}]_{g=0}$.

$$\omega_{0,2} = \frac{dy_1 dy_2}{(y_1 - y_2)^2} \qquad \mathrm{Res}_{X_1=0} \mathrm{Res}_{X_2=0} \frac{X_1^{-\lambda_1} X_2^{-\lambda_2}}{\lambda_1 \lambda_2} \omega_{0,2} = [H^\circ_{(\lambda_1,\lambda_2)}]_{g=0}$$

$\{P(x,y) = 0\} = \mathcal{S}$
$dx \wedge dy = dx' \wedge dy'$

$$\begin{tikzcd} \mathcal{S} \arrow[d, "x"'] \arrow[d, bend left, "x'"] \\ \mathbb{C} \end{tikzcd}$$

$$[\omega_{g,n}] \xleftrightarrow{DH^c} [\omega'_{g,n}]$$

$$\begin{tikzcd} \omega_{g,n} \arrow[rr, leftrightarrow] & & \omega'_{g,n} \\ \omega_{g,n,0} \arrow[u] & & \omega_{g,0,n} \arrow[u] \\ & \omega_{g,n,m} & \end{tikzcd}$$

$[\omega_{g,n,m}]$ only depends on $n+m$.

Prediction (Bouchard-Mariño) :

- qualitative part : exist $\omega_{g,n} \in H^0(K_{\mathcal{S}}(*A)^{\otimes n}, \mathcal{L}^n)^{\mathfrak{S}_n}$
  $A = \{dX = 0\} \subset \mathcal{S}$
  such that $[H^\circ_\lambda]_g = \mathrm{Res}_{X_1=0} \dots \mathrm{Res}_{X_n=0} \prod_{i=1}^{n} \frac{X_i^{-\lambda_i}}{\lambda_i} \omega_{g,n}$
- quantitative : $\omega_{g,n}$ governed by Eynard-Orantin topological recursion.

proved Eynard, Mulase, Safnuk 11

In brief $\quad \mathcal{S}$ smooth curve
x, y meromorphic f^n
A = zeroes of dx.
simple, disjoint from zeros of dy

$\xrightarrow{TR} \omega_{g,n}$

$$\omega_{g,n}(z_1, \dots, z_n) = \sum_{a \in A}{}' \mathrm{Res}_{z=a} \frac{\frac{1}{2}\int_{\sigma_a(z)}^{z} \omega_{0,2}(\cdot, z_1)}{(y(z) - y(\sigma_a(z)))dx(z)} \mathcal{E}_{g,n}(z, \sigma_a(z), z_2, \dots, z_n)$$

where $x \circ \sigma_a = x$ local involution near $a \in A$

$$\mathcal{E}_{g,n}(z, z', z_2, \dots, z_n) = \omega_{g-1,n+1}(z, z', z_2, \dots, z_n) + \sum_{\substack{h+h'=g \\ J \cup J' = \{z_2, \dots, z_n\}}}^{\text{no } \omega_{0,1}} \omega_{h,1+|J|}(z, J)\, \omega_{h',1+|J'|}(z', J')$$

only terms with smaller Euler characteristic.

Recursion.

④ Double Hurwitz # ($\mu \neq \phi$.)

- Enough to $\forall i\ \mu_i \leq d$
- Johnson-Pandharipande-Tseng (11).

$\overline{\mathcal{M}}_{g,a_1 \dots a_n}(\mathbb{P}^1[d], \lambda)$ $\qquad a_1, \dots, a_n \in \{0, \dots, d-1\}$
$\qquad \lambda$ = relative condition at 0
$d \mid D - \sum a_i \qquad D = |\lambda| \qquad a_1, \dots, a_n$ monodromies

$$\mathbb{E}^U \subset \mathbb{E} \to \overline{\mathcal{M}}_{g,a}(\mathbb{P}^1[d], \lambda) \xrightarrow{\varepsilon} \overline{\mathcal{M}}_{g,n} \qquad \Lambda^{[d]}_{g,a_1,\dots,a_n} = \varepsilon_*\left[\sum (-1)^h c_h(\mathbb{E}^{U\vee})\right] \leftrightarrow \text{Chiodo class}$$

generator of $\mathbb{Z}/d\mathbb{Z}$ acts by $\exp(\frac{2i\pi}{d})$ $\qquad$ Hodge bundle

Def $\quad \cdot\ \mu \vdash D \quad \max \mu_i \leq d \qquad \mu' = \mu$ without its parts of size d

μ
μ̌'
complement

- μ' is nonnegative if $D - |\mu'| \geq 0$.
- μ' is bounded if $\max_{i \neq j} (\check{\mu}'_i + \check{\mu}'_j) \geq d$

Thm (JPT, 11) ① If μ' is nonnegative and bounded

$$[H^\circ_{\lambda,\mu}]_g = d^{2g-2+n+\ell(\mu)} \prod_{i=1}^{n} \frac{(\lambda_i/d)^{\lfloor \lambda_i/d \rfloor}}{\lfloor \lambda_i/d \rfloor!} \int_{\overline{\mathcal{M}}_{g,n}} \frac{\Lambda^{[d]}_{g,-\bar{\lambda},\mu'}}{\prod_{i=1}^n (1 - \frac{\lambda_i}{d}\psi_i)}$$

$-\bar{\lambda}_i$ = unique integer in $\{0, \dots, d-1\}$ such that $-\lambda_i = -\bar{\lambda}_i$ mod d.

② if bounded, negative, the RHS = 0 (vanishing).

Thm (BDKLM, 20)

① $d \geq 1$, $q_1, \dots, q_d \in \mathbb{C}$ $\quad q_d \neq 0$ $\quad Q(z) = \sum q_j z^j$.

$$\begin{cases} X = z e^{-Q(z)} & x = \ln X \\ y = Q(z) & \\ \omega_{0,2} = \frac{dz_1 dz_2}{(z_1 - z_2)^2} & \end{cases} \xrightarrow{TR} (\omega_{g,n})_{g,n}$$

$$\mathrm{Res}_{X_1=0} \dots \mathrm{Res}_{X_n=0} \prod_{i=1}^{n} \frac{X_i^{-\lambda_i}}{\lambda_i} \omega_{g,n} = \sum_{\mu \vdash |\lambda|} \frac{\prod_{i=1}^{\ell(\mu)} q_{\mu_i}}{\# \mathrm{Aut}\,\mu} [H^\circ_{\lambda,\mu}]_g$$

q_j $j = 1, \dots, d$
interpolate between
d-orbifold
and double H.

② ELSV-like formula.

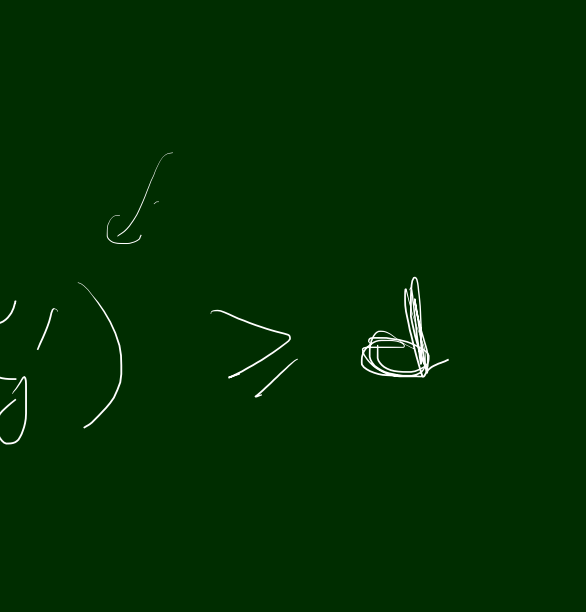
$$[H^\circ_{\lambda,\mu}]_g = d^{2g-2+n+\ell(\mu)} \prod_{i=1}^{n} \frac{(\lambda_i/d)^{\lfloor \lambda_i/d \rfloor}}{\lfloor \lambda_i/d \rfloor!} \times \left\{ \sum_{m=0}^{\ell(\mu')} \frac{(-1)^{\ell(\mu')-m}}{m!} \sum_{\substack{\rho^1 \sqcup \dots \sqcup \rho^m = \check{\mu}' \\ \max_j \rho^i_j \leq d-1}} \frac{\prod_{i=1}^{m} \left[\frac{d - |\rho^i|}{d}\right]_{\ell(\rho^i)-1}}{\# \mathrm{Aut}\,\rho'} \int_{\overline{\mathcal{M}}_{g,n+m}} \frac{\Lambda^{[d]}_{-\bar{\lambda}, d-|\rho^1|, d-|\rho^2|, \dots}}{\prod_{i=1}^{n} (1 - \frac{\lambda_i}{d}\psi_i)} \right\}$$

$$[x]_\ell = x(x+1) \dots (x+\ell-1).$$

if $\mu' = \phi$ ($\mu = (d, \dots, d)$) $\qquad m = 0$ only term $\quad$ d-orbifold Hurwitz # ($\subseteq$ JPT).

if $\mu' \neq \phi$ $\qquad m = 1, \dots, \ell(\mu')$.

③ if $|\lambda| - |\mu'| < 0$ (negative) RHS vanish



- This sum always contain $m = \ell(\mu')$ and this term is exactly JPT RHS
- In general : more terms but if μ' is bounded $\max_{i \neq j} (\check{\mu}'_i + \check{\mu}'_j) \geq d$ only the JPT term.

- In JPT : bubbling phenomena ⟹ for not bounded μ' boundary strata contributions,
  Can one get combinatorics of boundary strata ??